

EXISTENCE, UNIQUENESS AND MULTIPLICITY OF ROTATING FLUXON WAVES IN ANNULAR JOSEPHSON JUNCTIONS

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ABSTRACT. We prove that the equation modelling an annular Josephson junction has a rotating fluxon wave solution for all values of the parameters. We also obtain results on uniqueness of the rotating fluxon wave in some parameter regimes, and on multiplicity of rotating fluxon waves in other parameter regimes.

1. INTRODUCTION

The equation modelling a long Josephson junction is [7]

$$(1) \quad \phi_{tt} + \alpha\phi_t + \sin(\phi) = \phi_{xx} + \beta\phi_{xxt} + \gamma,$$

where $\alpha > 0, \beta > 0, \gamma > 0$ (α, β describe dissipative effects, while γ describes the applied bias current). In the case of annular geometry the periodicity condition

$$(2) \quad \phi(x + L, t) = \phi(x, t) + 2\pi,$$

is imposed, where L is the circumference of the junction.

Definition 1. A *rotating fluxon wave* is a solution of (1), (2) of the form

$$(3) \quad \phi(x, t) = \theta(kx + \omega t),$$

where the function $\theta(z)$ satisfies

$$(4) \quad \theta(z + 2\pi) = \theta(z) + 2\pi \quad \forall z \in \mathbb{R}.$$

Clearly the assumption that (3) satisfies (2) implies that

$$(5) \quad k = \frac{2\pi}{L},$$

while the frequency ω is an unknown to be found as part of the solution (in Proposition 5 we show that necessarily $\omega > 0$). We note that the time-period T of rotation of the wave around the annulus is given by $T = \frac{2\pi}{\omega}$.

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Substituting (3) into (1) we obtain

$$(6) \quad \beta\omega k^2 \theta'''(z) + (k^2 - \omega^2)\theta''(z) - \alpha\omega\theta'(z) - \sin(\theta(z)) + \gamma = 0,$$

so that rotating fluxon waves are given by solutions $(\omega, \theta(z))$ of (6), with $\theta(z)$ satisfying (4).

Our aim in this paper is to study the questions of existence and uniqueness/multiplicity of rotating fluxon waves. We shall obtain existence of a rotating fluxon wave for all values of the parameters:

Theorem 2. *For any $L > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, (1), (2) has a rotating fluxon wave solution (3), with $\omega > 0$.*

To prove Theorem 2 and our other results, we use methods of nonlinear functional analysis, reducing the problem to a fixed-point equation depending on a parameter (ω), to be solved by means of Leray-Schauder theory, coupled with an auxiliary scalar-valued equation, which we analyze. We have used a similar technique to prove existence of travelling waves of the *discrete* damped and dc-driven sine-Gordon equation [5]. The details of the analysis differ, and in the discrete sine-Gordon case we could not prove existence for all parameter values (which reflects the phenomenon of ‘pinning’ in the discrete case) - while here we are able to do so.

Several researchers have studied long Josephson junctions, and annular ones in particular, experimentally, [2, 8, 9], numerically [1, 6] and analytically [3, 4, 6, 7].

Note that in the case $\beta = 0$ (known as the damped dc-driven sine-Gordon equation), (6) reduces to the damped dc-driven pendulum equation which is amenable to phase-plane analysis and is thus well understood, and the existence of rotating fluxon waves for all $\alpha > 0, \gamma > 0$ is known (see [10], Section 2.2). The situation for $\beta > 0$ is different. Most existing analytical studies assume that either α, β and γ are small, so that the equation is treated as a (singular) perturbation of the sine-Gordon equation (the case $\alpha = 0, \beta = 0, \gamma = 0$ of (1)), or that β is small so that the equation is a perturbation of the damped dc-driven sine-Gordon equation. An exception is [6], which studies travelling fluxon waves of kink type on an infinite line, that is, solutions of (1) of the form (3) with (4) replaced by $\theta(+\infty) - \theta(-\infty) = 2\pi$. Existence results for such waves are proved by a geometric analysis of the corresponding three-dimensional phase-space, and a very intricate structure of solutions is revealed. Here, as we noted, our approach is functional-analytic rather than geometric, leading to the general existence result of Theorem 2.

Our method also allows us to derive some further information about the set of rotating fluxon waves. In the physical literature, a common and useful way to describe the behavior of a system is the ‘I-V characteristic’ - in this case the $\gamma - \omega$ (bias current-frequency) characteristic:

Definition 3. The *bias-frequency characteristic* is the set $\chi \subset (0, \infty) \times (0, \infty)$ of all pairs (ω, γ) for which (1),(2) has a rotating fluxon wave (3), with frequency ω .

The set χ depends, of course, on the parameters L, α, β . The following theorem shows that the set χ is ‘large’.

Theorem 4. *For any $L > 0, \alpha > 0, \beta > 0$, the bias-frequency characteristic χ has a subset $\chi_0 \subset \chi$ which has the following properties:*

- (i) χ_0 is connected.
- (ii) For any $\omega > 0$ there exists $\gamma > 0$ with $(\omega, \gamma) \in \chi_0$.
- (iii) For any $\gamma > 0$ there exists $\omega > 0$ with $(\omega, \gamma) \in \chi_0$.
- (iv) $(0, 0) \in \overline{\chi_0}$.
- (v) $\chi \setminus \chi_0$ is a bounded set.

Note that Theorem 2 follows at once from part (iii) of Theorem 4.

We shall also obtain some bounds on the frequency ω of rotating fluxon waves (see section 2):

Proposition 5. *For any rotating fluxon wave of (1), (2), we have $\omega > 0$ and*

$$(7) \quad \frac{\gamma - 1}{\alpha} \leq \omega \leq \frac{\gamma}{\alpha}.$$

We note that since we are assured that $\omega > 0$, the lower bound in (7) is nontrivial only when $\gamma > 1$.

The above results are completely general in that they hold for all values of the parameters. We also obtain some further results under some restrictions on the parameters. While we do not expect our conditions on the parameters, which are imposed by our methods of proof, to be sharp, these more restricted results expose some of the phenomena which can occur and point the way to further investigation (analytical or numerical) by raising various questions which remain unresolved.

We do not know whether in general we can take $\chi_0 = \chi$ in Theorem 4, or in other words whether the characteristic χ is connected. In Section 3 we shall provide a positive answer to this question, and also obtain further information about the bias-frequency characteristic χ , for a certain subset of the space of parameters L, α, β :

Theorem 6. *Assume $L > 0, \alpha > 0, \beta > 0$. If*

$$(8) \quad k = \frac{2\pi}{L} > 1$$

and

$$(9) \quad \alpha + \beta k^2 \geq \sqrt{2}k$$

then the bias-frequency characteristic χ can be represented in the form

$$(10) \quad \chi = \{(\omega, \overline{\gamma}(\omega)) \mid \omega > 0\}$$

where $\bar{\gamma} : [0, \infty) \rightarrow [0, \infty)$ is continuous and real-analytic in $(0, \infty)$, and satisfies

$$(11) \quad \bar{\gamma}(0) = 0,$$

$$(12) \quad \lim_{\omega \rightarrow +\infty} \bar{\gamma}(\omega) = +\infty.$$

In particular, χ is connected, and for a given bias $\gamma > 0$ the frequencies of rotating fluxon waves are the solutions ω of the equation

$$(13) \quad \bar{\gamma}(\omega) = \gamma.$$

Let us note that in the case $\beta = 0$, part of the above theorem is *not* true: while χ is connected and can be parameterized as in (10) (for any $k > 0$, $\alpha > 0$; see [10], sec. 2.2), the function $\bar{\gamma}$ is *not* analytic in the case $\beta = 0$, in fact it has a cusp at $\omega = k$ (see [10], fig. 2), and the value $\omega = k$ separates the ‘low-voltage’ and ‘high-voltage’ regions. Theorem 6 shows that the term ϕ_{xxt} in (1) has a smoothing effect on the curve χ (at least in the case that (8),(9) hold). When we prove Theorem 6 we will point out why the assumption $\beta > 0$ is crucial, which leads to this difference between the damped dc-forced sine-Gordon equation and (1).

A natural question is whether the rotating fluxon wave whose existence is guaranteed by Theorem 2, is unique (for given L, α, β, γ). In Section 4 we obtain a uniqueness result in the case that γ is sufficiently large.

Theorem 7. *Fix any $L > 0$, $\alpha > 0$, $\beta > 0$. Then for sufficiently large $\gamma > 0$, there is a **unique** rotating fluxon wave.*

On the other hand, in Section 5 we show that uniqueness does not hold in general, by proving that for certain parameter ranges there exist at least three rotating fluxon waves with different frequencies.

Theorem 8. *Assume (8), and replace (9) by the stronger assumption*

$$(14) \quad \beta k \geq \sqrt{2}.$$

*Then there exists a value $\gamma_0 > 0$ such that given any $0 < \epsilon < \frac{1}{2}\gamma_0$, there exists $\alpha_0 > 0$ so that for any $\alpha \in (0, \alpha_0)$ and any $\gamma \in (\epsilon, \gamma_0 - \epsilon)$, (1), (2) has **at least** three rotating fluxon solutions with distinct frequencies.*

We can expect this multiplicity of rotating fluxon waves to lead to bistability, jump and hysteresis phenomena as the bias γ is varied. We do not know whether the result of Theorem 8 is valid for all values of the parameters L, α, β , or whether, on the contrary, there are some values of these parameters for which there is a *unique* rotating fluxon wave for *all* $\gamma > 0$.

We mention here that all our results and proofs in this paper remain valid, with minor changes, if the nonlinearity $\sin(\phi)$ in (1) is replaced by $p(\phi)$, where p is an arbitrary C^1 2π -periodic function, satisfying $\int_0^{2\pi} p(\theta) d\theta = 0$.

An important issue which we do not address here is the question of stability. For example: is it true that for any value of $\alpha > 0, \beta > 0, \gamma > 0$ there exists at least one rotating fluxon wave of (1) which is at least locally asymptotically stable?

2. EXISTENCE OF ROTATING FLUXON WAVES

In this section we prove Theorem 4, which in particular implies Theorem 2. To do so we first recast our problem into a functional-analytic form, so that we can apply Leray-Schauder theory.

Multiplying both sides of (6) by $\theta'(z)$ and integrating over $[0, 2\pi]$, taking (4) into account, we obtain

$$\omega \left[\beta k^2 \int_0^{2\pi} (\theta''(z))^2 dz + \alpha \int_0^{2\pi} (\theta'(z))^2 dz \right] = 2\pi\gamma.$$

By our assumption that $\alpha > 0, \beta > 0, \gamma > 0$ this implies that $\omega > 0$, as stated in Proposition 5, and thus we may set

$$(15) \quad \lambda = \frac{1}{\omega}.$$

We also set

$$(16) \quad \theta(z) = z + u(z),$$

with $u(z)$ satisfying

$$(17) \quad u(z + 2\pi) = u(z) \quad \forall z \in \mathbb{R},$$

so that we can rewrite (6) as

$$(18) \quad \beta k^2 u'''(z) + \left(\lambda k^2 - \frac{1}{\lambda} \right) u''(z) - \alpha u'(z) - \lambda \sin(z + u(z)) + \lambda\gamma - \alpha = 0.$$

Rotating fluxon solutions of (1), (2) thus correspond to pairs $(\lambda, u(z))$ with $\lambda > 0$ and $u(z)$ satisfying (17), (18).

By integrating (18) over $[0, 2\pi]$, taking (17) into account, we obtain

$$(19) \quad \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds = \gamma - \frac{\alpha}{\lambda},$$

so that we can rewrite (18) as

$$(20) \quad \begin{aligned} & \beta k^2 u'''(z) + \left(\lambda k^2 - \frac{1}{\lambda} \right) u''(z) - \alpha u'(z) \\ & - \lambda \sin(z + u(z)) + \lambda \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds = 0. \end{aligned}$$

We note that if $(\lambda, u(z))$ is a solution to our problem then so is $(\lambda, u(z+c)+c)$ for any constant c (this of course corresponds to the time-invariance of (1)),

hence by adjusting c we may assume that

$$(21) \quad \int_0^{2\pi} u(s) ds = 0.$$

We thus seek pairs $(\lambda, u(z))$, satisfying (17), (19), (20) and (21).

Multiplying (20) by $1 + u'(z)$ and integrating over $[0, 2\pi]$ we obtain

Lemma 9. *Any solution (λ, u) of (17), (20) and (21) satisfies*

$$(22) \quad \lambda \int_0^{2\pi} \sin(s + u(s)) ds = \beta k^2 \int_0^{2\pi} (u''(s))^2 ds + \alpha \int_0^{2\pi} (u'(s))^2 ds.$$

In particular, together with (19), (22) implies that

$$\lambda \gamma \geq \alpha,$$

which gives the upper bound of (7). Also, noting that the value of the integral on the left-hand side of (19) is at most 1, we have

$$(23) \quad \gamma - 1 \leq \frac{\alpha}{\lambda},$$

which implies the lower bound in (7). We have thus proven Proposition 5.

We denote by X, Y, Z the Banach spaces of real-valued functions

$$X = \left\{ u \in H^3[0, 2\pi] \mid u^{(i)}(0) = u^{(i)}(2\pi), \ 0 \leq i \leq 2, \ \int_0^{2\pi} u(s) ds = 0 \right\},$$

$$Y = \left\{ u \in H^1[0, 2\pi] \mid \int_0^{2\pi} u(s) ds = 0 \right\},$$

$$Z = \left\{ u \in L^2[0, 2\pi] \mid \int_0^{2\pi} u(s) ds = 0 \right\},$$

with the norms

$$\|u\|_X = \left(\frac{1}{2\pi} \int_0^{2\pi} (u'''(s))^2 ds \right)^{\frac{1}{2}},$$

$$\|u\|_Y = \left(\frac{1}{2\pi} \int_0^{2\pi} (u'(s))^2 ds \right)^{\frac{1}{2}},$$

$$\|u\|_Z = \left(\frac{1}{2\pi} \int_0^{2\pi} (u(s))^2 ds \right)^{\frac{1}{2}}.$$

Assuming $v \in H^1[0, 2\pi]$ with $\int_0^{2\pi} v(s) ds = 0$, developing v in a Fourier series and using the Parseval identity one obtains

$$(24) \quad \int_0^{2\pi} (v(s))^2 ds \leq \int_0^{2\pi} (v'(s))^2 ds,$$

known as Wirtinger's inequality, implying the following inequalities which we will often use:

$$\|u\|_Y \leq \|u\|_X \quad \forall u \in X, \quad \|u\|_Z \leq \|u\|_Y \quad \forall u \in Y.$$

We define a family of linear mappings $L_\lambda : X \rightarrow Z$ ($\lambda \neq 0$) by

$$(25) \quad L_\lambda(u) = \beta k^2 u'''(z) + \left(\lambda k^2 - \frac{1}{\lambda} \right) u''(z) - \alpha u'(z)$$

and a nonlinear mapping $N : Z \rightarrow Z$ by

$$(26) \quad N(u) = \sin(z + u(z)) - \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds.$$

Solving (17), (20), (21) is equivalent to solving

$$(27) \quad L_\lambda(u) = \lambda N(u), \quad u \in X,$$

so that our problem reduces to: find $(\lambda, u) \in (0, \infty) \times X$ such that (27) and (19) hold (we note that, by a simple bootstrap argument, any solution $u \in X$ of (27) is in fact C^∞). In other words, defining the nonlinear functional $\Phi : (0, \infty) \times Z \rightarrow \mathbb{R}$ by

$$(28) \quad \Phi(\lambda, u) = \frac{\alpha}{\lambda} + \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds,$$

we need to solve the pair of equations (27) and

$$(29) \quad \Phi(\lambda, u) = \gamma,$$

for the unknowns $(\lambda, u) \in (0, \infty) \times X$.

We derive some estimates on L_λ which will be useful. Let $u \in X$ and write u as a Fourier series

$$u(z) = \sum_{l \neq 0} a_l e^{ilz}.$$

Noting that

$$L_\lambda(e^{ilz}) = - \left[(l\alpha + \beta k^2 l^3) i + l^2 (\lambda k^2 - \lambda^{-1}) \right] e^{ilz},$$

so that

$$L_\lambda^{-1}(e^{ilz}) = - \left[(l\alpha + \beta k^2 l^3) i + l^2 (\lambda k^2 - \lambda^{-1}) \right]^{-1} e^{ilz},$$

we obtain

$$\begin{aligned}
\|L_\lambda^{-1}(u)\|_Y &= \left\| \sum_{l \neq 0} a_l \left[(l\alpha + \beta k^2 l^3)i + l^2(\lambda k^2 - \lambda^{-1}) \right]^{-1} e^{ilz} \right\|_Y \\
&= \left(\sum_{l \neq 0} l^2 |a_l|^2 \left[(l\alpha + \beta k^2 l^3)^2 + l^4(\lambda k^2 - \lambda^{-1})^2 \right]^{-1} \right)^{\frac{1}{2}} \\
&= \left(\sum_{l \neq 0} |a_l|^2 \left[(\alpha + \beta k^2 l^2)^2 + l^2(\lambda k^2 - \lambda^{-1})^2 \right]^{-1} \right)^{\frac{1}{2}} \\
&\leq \left[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2 \right]^{-\frac{1}{2}} \left[\sum_{l \neq 0} |a_l|^2 \right]^{\frac{1}{2}} \\
&= \left[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2 \right]^{-\frac{1}{2}} \|u\|_Z.
\end{aligned}$$

so that

$$(30) \quad \|L_\lambda^{-1}\|_{Z,Y} \leq \left[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2 \right]^{-\frac{1}{2}}.$$

In particular (dropping the first term on the right-hand side of (30))

$$(31) \quad \|L_\lambda^{-1}\|_{Z,Y} \leq \frac{\lambda}{|\lambda^2 k^2 - 1|}.$$

We can also obtain

$$\begin{aligned}
\|L_\lambda^{-1}(u)\|_X &= \left(\sum_{l \neq 0} l^6 |a_l|^2 \left[(l\alpha + \beta k^2 l^3)^2 + l^4(\lambda k^2 - \lambda^{-1})^2 \right]^{-1} \right)^{\frac{1}{2}} \\
&\leq \frac{\lambda}{|\lambda^2 k^2 - 1|} \left(\sum_{l \neq 0} l^2 |a_l|^2 \right)^{\frac{1}{2}} = \frac{\lambda}{|\lambda^2 k^2 - 1|} \|u\|_Y,
\end{aligned}$$

so that

$$(32) \quad \|L_\lambda^{-1}\|_{Y,X} \leq \frac{\lambda}{|\lambda^2 k^2 - 1|}.$$

Defining $F : (0, \infty) \times Z \rightarrow Z$ by

$$(33) \quad F(\lambda, u) = \lambda L_\lambda^{-1} \circ N(u),$$

we may rewrite (27) as

$$(34) \quad u = F(\lambda, u), \quad u \in Z.$$

We note that since L_λ^{-1} maps Z onto X , any solution $u \in Z$ of (34) is in fact in X .

Since $\|\sin(z + u(z))\|_{L^2} \leq 1$ and $N(u)$ is the orthogonal projection of $\sin(z + u(z)) \in L^2[0, 2\pi]$ into Z , we have

$$(35) \quad \|N(u)\|_Z \leq 1 \quad \forall u \in Z.$$

Using (35) and (30), we have the bound:

$$(36) \quad \|F(\lambda, u)\|_Z \leq \|F(\lambda, u)\|_Y \leq \lambda \left[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2 \right]^{-\frac{1}{2}} \quad \forall u \in Z, \lambda > 0.$$

We extend the definition of $F(\lambda, u)$ by setting $F(0, u) = 0$ for all $u \in Z$ (this is necessary because L_λ is undefined for $\lambda = 0$). The bound (36) shows that the extension is continuous, and moreover that the mapping F takes bounded sets in $[0, \infty) \times Z$ into bounded sets in Y , hence (since the embedding of Y into Z is compact) into compact sets in Z , so that $F : [0, \infty) \times Z \rightarrow Z$ is a compact mapping.

We define

$$\Sigma = \{(\lambda, u) \in (0, \infty) \times Z \mid u = F(\lambda, u)\}.$$

The solutions of (27), (29), are the solutions of the equation

$$(37) \quad \Phi(\lambda, u) = \gamma, \quad (\lambda, u) \in \Sigma.$$

In other terms, recalling (15), the bias-frequency characteristic χ defined in the introduction is given by

$$(38) \quad \chi = \{(\omega, \Phi(\omega^{-1}, u)) \mid (\omega^{-1}, u) \in \Sigma\}.$$

We will now apply the following version of the Leray-Schauder continuation principle (see, *e.g.*, [11], Theorem 14.D):

Theorem 10. *Let Z be a Banach space and let $F : [0, \infty) \times Z \rightarrow Z$ be a continuous and compact mapping, with*

$$F(0, u) = 0 \quad \forall u \in Z.$$

Define $\Sigma \subset (0, \infty) \times Z$ by

$$\Sigma = \{(\lambda, u) \in (0, \infty) \times Z \mid u = F(\lambda, u)\}.$$

Then there exists an unbounded connected set $\Sigma_0 \subset \Sigma$ with $(0, 0) \in \overline{\Sigma}_0$.

Since the mapping F that we have defined satisfies the hypotheses of Theorem 10, we have a set Σ_0 as in the Theorem.

We define $\chi_0 \subset \chi$ by

$$(39) \quad \chi_0 = \{(\omega, \Phi(\omega^{-1}, u)) \mid \omega > 0, (\omega^{-1}, u) \in \Sigma_0\},$$

and we will show that χ_0 has all the properties claimed in Theorem 4.

Since Φ is continuous and Σ_0 is connected, χ_0 is connected, so we have (i) of Theorem 4.

Part (ii) of Theorem 4 follows from the following

Lemma 11. *Let Σ_0 be as in Theorem 10. For any $\lambda > 0$ there exists $u \in Z$ such that $(\lambda, u) \in \Sigma_0$.*

PROOF: Since from (34) and (36) we have

$$(40) \quad (\lambda, u) \in \Sigma \Rightarrow \|u\|_Z \leq \lambda \frac{1}{\beta k^2 + \alpha},$$

we conclude that the unboundedness of Σ_0 is in the λ -direction, that is, for any $M > 0$ there exists $(\lambda, u) \in \Sigma_0$ with $\lambda > M$. In other words, denoting by $\Lambda \subset (0, \infty)$ the projection of Σ_0 to the λ -axis,

$$\Lambda = \{\lambda > 0 \mid \exists u \in Z \text{ such that } (\lambda, u) \in \Sigma_0\},$$

we have that Λ contains arbitrarily large values. Since $(0, 0) \in \bar{\Sigma}_0$, Λ also contains arbitrary small positive values. Since Σ_0 , hence Λ , is connected, we conclude that $\Lambda = (0, \infty)$. \square

To prove that χ_0 satisfies property (iii) of Theorem 4 (and hence also Theorem 2), we need to show that

$$(41) \quad (0, \infty) \subset \Phi(\Sigma_0).$$

Since Σ_0 is connected and Φ is continuous, $\Phi(\Sigma_0)$ is also connected, so that to prove (41) it suffices to show that Φ takes arbitrarily large and arbitrarily small positive values on Σ_0 . Since the integral in the definition (28) of Φ is bounded by 1, it is immediate that

$$(42) \quad \lim_{\lambda \rightarrow 0+, (\lambda, u) \in \Sigma} \Phi(\lambda, u) = +\infty,$$

so that Φ takes arbitrarily large values on Σ_0 . We shall prove

Lemma 12.

$$(43) \quad \lim_{\lambda \rightarrow +\infty, (\lambda, u) \in \Sigma} \Phi(\lambda, u) = 0.$$

This implies that Φ takes arbitrarily small positive values on Σ_0 , completing the proof of (41), and thus of part (iii) of Theorem 4 holds.

PROOF OF LEMMA 12: Applying Wirtinger's inequality (24) with $v = u'$, Lemma 9 implies

$$(44) \quad (\lambda, u) \in \Sigma \Rightarrow \lambda \frac{1}{2\pi} \left| \int_0^{2\pi} \sin(s + u(s)) ds \right| \leq (\beta k^2 + \alpha) \frac{1}{2\pi} \int_0^{2\pi} (u'(s))^2 ds.$$

From (34) and (31) we obtain

$$(45) \quad (\lambda, u) \in \Sigma \Rightarrow \|u\|_Y \leq \frac{\lambda^2}{|\lambda^2 k^2 - 1|}.$$

(44) together with (45) implies

$$(\lambda, u) \in \Sigma \Rightarrow \left| \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds \right| \leq \frac{(\beta k^2 + \alpha) \lambda^3}{(\lambda^2 k^2 - 1)^2},$$

which implies that

$$(46) \quad \lim_{\lambda \rightarrow +\infty, (\lambda, u) \in \Sigma} \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds = 0,$$

hence we have (43). \square

To prove claim (iv) of Theorem 4, let $\omega_n = \frac{1}{n}$ ($n \geq 1$), and note that by part (ii) of the Theorem, there exists, for each n , $\gamma_n > 0$ such that $(\omega_n, \gamma_n) \in \chi_0$. By the definition of χ_0 there exists $u_n \in \Sigma_0$ such that $\gamma_n = \Phi(\omega_n^{-1}, u_n)$. But by Lemma 12 we have $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$. Hence we have $(\omega_n, \gamma_n) \rightarrow (0, 0)$, so we have $(0, 0) \in \overline{\chi_0}$.

To prove part (v) of Theorem 4, we will use the following two lemmas, which will also be used in the following sections.

Lemma 13. *If*

$$(47) \quad \lambda \left[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2 \right]^{-\frac{1}{2}} < 1$$

then $F(\lambda, \cdot) : Z \rightarrow Y$ (hence also $F(\lambda, \cdot) : Z \rightarrow Z$) is a contraction.

PROOF: Computing the Frechét derivative of $N : Z \rightarrow Z$ we have

$$N'(u)(v) = \cos(z + u(z))v(z) - \frac{1}{2\pi} \int_0^{2\pi} \cos(s + u(s))v(s) ds,$$

so that $N'(u)$ is the mapping from Z to $L^2[0, 2\pi]$ given by $v(z) \rightarrow \cos(z + u(z))v(z)$ followed by an orthogonal projection into the subspace Z , and this implies

$$(48) \quad \|N'(u)\|_{Z, Z} \leq 1 \quad \forall u \in Z.$$

From (30), (33) and (48) we conclude that

$$(49) \quad \|D_u F(\lambda, u)\|_{Z, Y} = \lambda \|L_\lambda^{-1} N'(u)\|_{Z, Y} \leq \lambda \left[(\alpha + \beta k^2)^2 + (\lambda k^2 - \lambda^{-1})^2 \right]^{-\frac{1}{2}} \quad \forall u \in Z,$$

which, by (47), completes the proof. \square

Lemma 14. *Fixing $\alpha \geq 0$, $\beta > 0$, there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$ the equation (34) has a **unique** solution, which we denote by u_λ . The mapping $\lambda \rightarrow u_\lambda$ from $[0, \lambda_0)$ into Z is continuous and is real-analytic for $\lambda \in (0, \lambda_0)$.*

PROOF: It is easy to see that we can find $\lambda_0 > 0$ such that (47) holds for $\lambda \in [0, \lambda_0)$, and thus by Lemma 13, for any $\lambda \in [0, \lambda_0)$ the mapping $F(\lambda, \cdot) : Z \rightarrow Z$ is a contraction, so that, by Banach's fixed-point Theorem, equation (34) has a **unique** solution, which we denote by u_λ . Real-analyticity of the mapping $\lambda \rightarrow u_\lambda$ follows from the real-analytic implicit function Theorem. \square

Returning to the proof of part (v) of Theorem 4, we note that since, by Lemma 11, for each λ there exists u such that $(\lambda, u) \in \Sigma_0$, and since the uniqueness in Lemma 14 implies that when $\lambda < \lambda_0$ such a u is unique in Σ , we must have

$$(50) \quad 0 < \lambda < \lambda_0, (\lambda, u) \in \Sigma \Rightarrow (\lambda, u) \in \Sigma_0.$$

Thus if we set $\omega_0 = \frac{1}{\lambda_0}$ we have by (50), (38), (39)

$$(51) \quad \omega > \omega_0, (\omega, \gamma) \in \chi \Rightarrow (\omega, \gamma) \in \chi_0.$$

Setting $\gamma_0 = \alpha\omega_0 + 1$ and using Proposition 5 and (51) we obtain

$$(52) \quad \gamma > \gamma_0, (\omega, \gamma) \in \chi \Rightarrow \omega > \omega_0, (\omega, \gamma) \in \chi \Rightarrow (\omega, \gamma) \in \chi_0,$$

and (51), (52) mean that

$$(\omega, \gamma) \in \chi \setminus \chi_0 \Rightarrow \omega \leq \omega_0, \gamma \leq \gamma_0,$$

so that $\chi \setminus \chi_0$ is bounded.

We note for later use that (22) implies that

Lemma 15.

$$(\lambda, u) \in \Sigma, \lambda > 0 \Rightarrow \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds > 0.$$

The strictness in the above inequality follows from the fact that, by (22), if we had equality this would imply $u = 0$, but it is easy to check that $(\lambda, 0) \notin \Sigma$ for $\lambda > 0$.

3. MORE ON THE BIAS-FREQUENCY CHARACTERISTIC

In this section we prove Theorem 6, which shows that for a certain open subset of the space of parameters L, α, β , the bias-frequency characteristic χ is a connected, real analytic curve, and can be represented as the graph of a function: $\gamma = \overline{\gamma}(\omega)$.

Lemma 16. *Assume that $\alpha \geq 0, \beta > 0$ and the inequalities (8) and (9) hold. Then for all $\lambda \geq 0$, the equation (34) has a **unique** solution, which we denote by u_λ , and therefore*

$$(53) \quad \Sigma = \Sigma_0 = \{(\lambda, u_\lambda) \mid \lambda \geq 0\}.$$

The mapping $\lambda \rightarrow u_\lambda$ from $[0, \infty)$ into Y is continuous, and real-analytic on $(0, \infty)$.

PROOF: We use Banach's fixed point Theorem. By Lemma 13, to show that $F(\lambda, \cdot)$ is a contraction from Z to itself for all $\lambda \geq 0$ it suffices to show that (47) holds for all $\lambda \geq 0$. After some manipulations, and setting $\mu = \lambda^2$, we see that this inequality is equivalent to

$$(54) \quad p(\mu) \equiv (k^4 - 1)\mu^2 + [(\alpha + \beta k^2)^2 - 2k^2]\mu + 1 > 0 \quad \forall \mu \geq 0.$$

The condition $k > 1$ ensures that the quadratic function $p(\mu)$ has a minimum, and we have $p(0) = 1 > 0$. Moreover the condition (9) ensures that $p'(0) \geq 0$, which implies p is increasing on $[0, \infty)$, so that (54) holds. We thus obtain a unique solution u_λ to (34). The fact that it is real-analytic in $\lambda \in (0, \infty)$ follows from the analytic implicit function Theorem and the fact that the function $\lambda \rightarrow L_\lambda^{-1}$ from $(0, \infty)$ to the space $B(Z)$ of bounded linear operators from Z to itself is real-analytic. \square

Let us note here that it is precisely the last statement in the above proof - the fact that $\lambda \rightarrow L_\lambda^{-1}$ is real-analytic, that breaks down when $\beta = 0$, and is responsible for the fact that in that case (*i.e.* the case of the damped, dc-forced sine-Gordon equation) the curve χ is *not* smooth (see [10], fig. 2). The reason that, when $\beta = 0$, $\lambda \rightarrow L_\lambda^{-1}$ is not real-analytic can be gleaned by recalling the definition (25): if $\beta = 0$ then at $\lambda = \frac{1}{k}$ the coefficient of the highest derivative in the operator L_λ vanishes.

Thus, under the assumptions of Lemma 16, we have

$$\chi = \chi_0 = \{(\omega, \Psi(\omega^{-1})) \mid \omega > 0\},$$

where $\Psi : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$(55) \quad \Psi(\lambda) = \Phi(\lambda, u_\lambda),$$

so defining $\overline{\gamma}(\omega) = \Psi(\omega^{-1})$ we have (10). The fact that $\overline{\gamma}(\omega) > 0$ for $\omega > 0$ follows from Lemma 15. (11) follows from Lemma 12, and (12) follows from (42). This concludes the proof of Theorem 6.

4. UNIQUENESS OF ROTATING FLUXON WAVES FOR LARGE BIAS CURRENT

We now prove Theorem 7. By Lemma 14, defining λ_0 as in Lemma 14, for any $\lambda \in [0, \lambda_0)$ the equation (34) has a **unique** solution, which we denote by u_λ . We define the function $\Psi : (0, \lambda_0) \rightarrow \mathbb{R}$ by (55), where Φ is the functional defined by (28), so that solutions of (37) with $\lambda \in (0, \lambda_0)$ correspond to solutions of

$$(56) \quad \Psi(\lambda) = \gamma.$$

We show below that there exists some $\lambda_1 \in (0, \lambda_0)$ such that

$$(57) \quad \lambda \in (0, \lambda_1) \Rightarrow \Psi'(\lambda) < 0.$$

We note that if γ is sufficiently large then, by (7), we have $\lambda \in (0, \lambda_1)$ for any solution of (56). Therefore (57) implies uniqueness of solutions of (56), hence uniqueness of the rotating fluxon wave, for sufficiently large γ , which is the content of Theorem 7.

To prove claim (57) we note that, since

$$\Psi(\lambda) = \frac{\alpha}{\lambda} + \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u_\lambda(s)) ds,$$

we have

$$\Psi'(\lambda) = -\frac{\alpha}{\lambda^2} + \frac{1}{2\pi} \int_0^{2\pi} \cos(s + u_\lambda(s)) D_\lambda u_\lambda(s) ds,$$

and using Cauchy's inequality

$$(58) \quad \Psi'(\lambda) \leq -\frac{\alpha}{\lambda^2} + \|D_\lambda u_\lambda\|_Z,$$

so to prove (57) it suffices to show that $\|D_\lambda u_\lambda\|_Z = o(\frac{1}{\lambda^2})$ as $\lambda \rightarrow 0+$. In fact we shall prove a much stronger statement:

Lemma 17. *We have*

$$(59) \quad \|D_\lambda u_\lambda\|_Z = O(\lambda) \quad \text{as } \lambda \rightarrow 0+.$$

PROOF: We differentiate the identity

$$L_\lambda(u_\lambda) = \lambda N(u_\lambda)$$

with respect to λ , obtaining

$$(60) \quad (D_\lambda L_\lambda)(u_\lambda) + L_\lambda(D_\lambda u_\lambda) = N(u_\lambda) + \lambda N'(u_\lambda)(D_\lambda u_\lambda).$$

We note that, because we assume $\lambda \in (0, \lambda_0)$, where λ_0 is defined as in Lemma 14, $\lambda L_\lambda^{-1} N'(u_\lambda)$ is a contraction from Z to itself so the inverse of $I - \lambda L_\lambda^{-1} N'(u_\lambda)$ is well defined as a mapping from Z to itself. Moreover we have, using (49),

$$(61) \quad \|[I - \lambda L_\lambda^{-1} N'(u_\lambda)]^{-1}\|_{Z,Z} \leq \frac{1}{1 - \lambda \|L_\lambda^{-1} N'(u_\lambda)\|_{Z,Z}} = O(1) \quad \text{as } \lambda \rightarrow 0+.$$

We can rewrite (60) in the form

$$D_\lambda u_\lambda = [I - \lambda L_\lambda^{-1} N'(u_\lambda)]^{-1} L_\lambda^{-1} [N(u_\lambda) - (D_\lambda L_\lambda)(u_\lambda)],$$

which implies

$$(62) \quad \|D_\lambda u_\lambda\|_Z \leq \|[I - \lambda L_\lambda^{-1} N'(u_\lambda)]^{-1}\|_{Z,Z} \left[\|L_\lambda^{-1} N(u_\lambda)\|_Z + \|L_\lambda^{-1} (D_\lambda L_\lambda)(u_\lambda)\|_Z \right].$$

Using (45) we obtain

$$\begin{aligned} \|N(u_\lambda)\|_Y &= \left(\frac{1}{2\pi} \int_0^{2\pi} \cos^2(s + u_\lambda(s)) (1 + u'_\lambda(s))^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{2\pi} \int_0^{2\pi} (1 + u'_\lambda(s))^2 ds \right)^{\frac{1}{2}} = [1 + \|u_\lambda\|_Y^2]^{\frac{1}{2}} \leq \left[1 + \left(\frac{\lambda^2}{\lambda^2 k^2 - 1} \right)^2 \right]^{\frac{1}{2}}, \end{aligned}$$

so that

$$(63) \quad \|N(u_\lambda)\|_Y = O(1) \quad \text{as } \lambda \rightarrow 0+.$$

From (32) and (63) we have

$$(64) \quad \|L_\lambda^{-1}N(u_\lambda)\|_Z \leq \|L_\lambda^{-1}N(u_\lambda)\|_X \leq \|L_\lambda^{-1}\|_{Y,X} \|N(u_\lambda)\|_Y = O(\lambda) \quad \text{as } \lambda \rightarrow 0+.$$

From the definition (25) of L_λ we have, for all $u \in X$

$$(D_\lambda L_\lambda)(u) = \left(k^2 + \frac{1}{\lambda^2}\right)u''(z),$$

hence

$$(65) \quad \|(D_\lambda L_\lambda)(u)\|_Y = \left(k^2 + \frac{1}{\lambda^2}\right)\|u\|_X \quad \forall u \in X.$$

From (64) we have

$$(66) \quad \|u_\lambda\|_X = \lambda \|L_\lambda^{-1}N(u_\lambda)\|_X = O(\lambda^2) \quad \text{as } \lambda \rightarrow 0+,$$

so using (65), (66)

$$(67) \quad \|(D_\lambda L_\lambda)(u_\lambda)\|_Y = O(1) \quad \text{as } \lambda \rightarrow 0+,$$

and together with (32)

$$(68) \quad \|L_\lambda^{-1}(D_\lambda L_\lambda)(u_\lambda)\|_Z \leq \|L_\lambda^{-1}(D_\lambda L_\lambda)(u_\lambda)\|_X = O(\lambda) \quad \text{as } \lambda \rightarrow 0+.$$

Using (61), (62), (64) and (68) we obtain (59). \square

5. MULTIPLICITY OF ROTATING FLUXON WAVES

In this section we prove Theorem 8, which shows that for some values of the parameters L, β , if $\alpha > 0$ is sufficiently small then there exists an interval of γ 's for which there exist at least three rotating fluxon waves.

Lemma 18. *Assume that $\beta > 0$ and (8), (14) hold. Then for any $\alpha \geq 0$, $\lambda \geq 0$, the equation (34) has a **unique** solution, which we denote by $u_{\alpha, \lambda}$. The mapping $(\alpha, \lambda) \rightarrow u_{\alpha, \lambda}$ from $[0, \infty) \times [0, \infty)$ into Z is continuous.*

As in the proof of Lemma 16, this follows from the Banach fixed point Theorem applied to (34). Note that (14) implies that (9) holds for all $\alpha \geq 0$.

Since in the following argument the dependence on α is important, we display it in the notation (of course $u_{\alpha, \lambda}$ also depends on the parameters k and β , but since we shall regard these as fixed we can suppress this dependence).

PROOF OF THEOREM 8: Under the conditions of Lemma 18 we can define $\Psi_\alpha(\lambda)$ as in (55), now defined for all $\lambda \in [0, \infty)$, so that rotating fluxon waves correspond to solutions of

$$(69) \quad \Psi_\alpha(\lambda) = \gamma, \quad \lambda > 0.$$

We write $\Psi_\alpha(\lambda)$ in the form (see (28))

$$(70) \quad \Psi_\alpha(\lambda) = \frac{\alpha}{\lambda} + \Upsilon_\alpha(\lambda),$$

where

$$\Upsilon_\alpha(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u_{\alpha,\lambda}(s)) ds.$$

We note some properties of the function $\Upsilon_\alpha(\lambda)$ (valid for any $\alpha \geq 0$). Since $u_{\alpha,0} = 0$ we have

$$(71) \quad \Upsilon_\alpha(0) = 0.$$

From (46) we obtain:

$$(72) \quad \lim_{\lambda \rightarrow +\infty} \Upsilon_\alpha(\lambda) = 0.$$

Lemma 15 implies

$$(73) \quad \Upsilon_\alpha(\lambda) > 0 \quad \forall \lambda > 0.$$

(71),(72),(73), imply that $\Upsilon_\alpha(\lambda)$ attains a positive global maximum at some $\lambda > 0$. In particular this is true for $\alpha = 0$, and we denote the global maximum of $\Upsilon_0(\lambda)$ by γ_0 , and assume that it is attained at $\lambda_0 > 0$:

$$(74) \quad \Upsilon_0(\lambda_0) = \gamma_0 = \max_{\lambda > 0} \Upsilon_0(\lambda).$$

We now fix an arbitrary $0 < \epsilon < \frac{1}{2}\gamma_0$. By (71),(72) we can choose $0 < \lambda_1 < \lambda_0$ and $\lambda_2 > \lambda_0$ so that

$$(75) \quad \Upsilon_0(\lambda_1) < \frac{\epsilon}{4}, \quad \Upsilon_0(\lambda_2) < \frac{\epsilon}{4}.$$

Since the dependence of $\Upsilon_\alpha(\lambda)$ on α is continuous, we can choose $\alpha_0 > 0$ sufficiently small so that

$$(76) \quad |\Upsilon_\alpha(\lambda) - \Upsilon_0(\lambda)| < \frac{\epsilon}{2} \quad \forall \alpha \in [0, \alpha_0], \lambda \in [0, \lambda_2].$$

From (74),(75) and (76) we obtain, for all $\alpha \in [0, \alpha_0]$,

$$(77) \quad \Upsilon_\alpha(\lambda_0) > \gamma_0 - \frac{\epsilon}{2},$$

$$(78) \quad \Upsilon_\alpha(\lambda_1) < \frac{3}{4}\epsilon, \quad \Upsilon_\alpha(\lambda_2) < \frac{3}{4}\epsilon.$$

Moreover we can choose $\alpha_0 > 0$ sufficiently small so that

$$(79) \quad \frac{\alpha_0}{\lambda_1} < \frac{\epsilon}{2}.$$

From (70) and (79) we then obtain

$$|\Psi_\alpha(\lambda) - \Upsilon_\alpha(\lambda)| < \frac{\epsilon}{2} \quad \forall \alpha \in [0, \alpha_0], \lambda \geq \lambda_1.$$

Together with (77),(78) this gives, for all $0 < \alpha < \alpha_0$,

$$(80) \quad \Psi_\alpha(\lambda_0) > \gamma_0 - \epsilon,$$

$$(81) \quad \Psi_\alpha(\lambda_1) < \epsilon,$$

$$(82) \quad \Psi_\alpha(\lambda_2) < \epsilon.$$

and by (70) we also have

$$(83) \quad \lim_{\lambda \rightarrow 0+} \Psi_\alpha(\lambda) = +\infty.$$

Thus by continuity of Ψ_α on $(0, \infty)$, if we take $\gamma \in (\epsilon, \gamma_0 - \epsilon)$ then by (83) and (81) the equation (69) has a solution $\lambda \in (0, \lambda_1)$, by (81) and (80) the equation (69) has a solution $\lambda \in (\lambda_1, \lambda_0)$, and by (80) and (82) the equation (69) has a solution $\lambda \in (\lambda_1, \lambda_2)$. We have thus found, under the stated assumptions, three solutions of (69), corresponding to three rotating fluxon waves with different frequencies. \square

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